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On Strongly Regular Graphs with Parameters (k,0,2) and Their Antipodal Double Covers

Nobuo NAKAGAWA(中川 暢夫)
Department of Mathematics, Faculty of Science and Technology, Kinki University, Higashi-Osaka, Osaka 577-8520, JAPAN

Abstract

Let Γ be a strongly regular graph with parameters $(k,\lambda,\mu)=(q^2+1,0,2)$ admitting $G(\cong PGL(2,q^2))$ as one point stabilizer for odd prime power q. We show that if G stabilizes a vertex ∞ of Γ and acts on $\Gamma_2(\infty)$ transitively, then $q=3$ holds and Γ is the Gewirtz graph. Moreover it is showen that an antipodal double cover whose diameter 4 of a strongly regular graph with parameters $(k,0,2)$ is reconstructed from a symmetric association scheme of class 6 with suitable parameters.

1 Introduction

We are interested in the classification problems of distance regular graphs with $b_2=1$. Let Γ be a distance regular graph with $b_2=1$ and valency $k>2$. If the diameter d of Γ is larger more than 4, then Γ is isomorphic to the dodecahedron ([3, pp.182]). In [1], M.Araya,A.Hiraki and A.Jurisic showed that if $d=4$,then Γ is an antipodal double cover of a strongly regular graph with parameters $(k,\lambda,\mu)=(n^2+1,0,2)$ for an integer n not divisible by four and if $d=3$, then Γ is an antipodal cover of a complete graph. Obviously an antipodal cover of a complete graph is a distance regular graph with $b_2=1$ if it's diameter is three.

The classification problems of antipodal covers of complete graphs are very difficult. Because the existence of an antipodal distance regular $(n-2)$-fold cover of the complete graph K_n claims the existence of a projective plane of order $(n-1)$ for an odd positive integer n, moreover an antipodal distance regular $(n-1)$-fold cover of K_n is equivalent to the existence of a Moore graph with diameter two and valency n ([6],[7]).

The strongly regular graphs with parameters $(k,\lambda,\mu)=(5,0,2)$ and $(k,\lambda,\mu)=(10,0,2)$ are known, the former one has an antipodal double cover with $d=4$, namely the Wells graph, the latter one(the Gewirtz graph) does not have an antipodal double cover with $d = 4$([3, pp.372]). The existence or nonexistence of strongly regular graphs with $(n^2+1,0,2)$ for $n \geq 5$ are not known up to date. We have studied these graphs.

2 Strongly regular graphs with $(q^2+1,0,2)$ admitting $PGL(2,q^2)$ for $q = p^e$

The following theorem is proved by using the character table of the association scheme corresponding to the permutation group $(O(3,q), O(3,q)/O^+(2,q))$ which W.M.Kwok gave in [5]. We note that $O(3,q) \cong \{\pm 1\} \times SO(3,q)$ and $SO(3,q) \cong PGL(2,q)$.

Theorem 2.1 *Let Γ be a strongly regular graph with parameters $(q^2+1,0,2)$ and G be a group isomorophic to $PGL(2,q^2)$ for an odd prime power q. If G acts on Γ as G stabilizes a vertex ∞ of Γ and G is transitive on $\Gamma_2(\infty)$, then $q = 3$ and Γ is the Gewirtz graph.*

Sketch of the proof)
Any two involutions of G are conjugate each other in G. We denote the centralizer of an involution z in G by H. Character table of association scheme $\mathcal{X}$ corresponding to the permutation group $(G, G/H)$ is given from Kwok's results. Then we obtain sevral informations concerning eigenvalues and their multiplicities of the graph $\Gamma_2(\infty)$ admitting G as a transitive automorphism group from the character table of $\mathcal{X}$.

Comparing these informations with eigenvalues and their multiplicities of $\Gamma_2(\infty)$ as the second neighbourhood of a strongly regular graph with parametars $(q^2+1,0,2)$, we can lead a contradiction if $q > 3$.

3 Reconstruction of the graph Γ and the antipodal double cover Γ^* of Γ

Let Γ be a strongly regular graphs with parameters $(k,0,2)$. In this section we study about the structure of the second neighbourhood of Γ and antipodal double covers of them with $d = 4$. E.R.van.Dam and A.Munemasa proved the following theorem 3.1 independently. ([4, pp.13-14],[8])

Theorem 3.1 *Let Γ be a strongly regular graph with $\lambda = 0$, $\mu = 2$ and degree k with $k > 5$. Then the second neighbourhood of Γ with respect to any vertex generates a 3-class association scheme. Furthermore any scheme with the same parameters can be constructed in this way from a strongly regular graph with the same parameters as Γ.*

The intersection numbers $p^j_{h,i}$ of the association scheme of theorem 3.1 are the following. Let $B_h (0 \leq h \leq 3)$ be the intersection matrices which $(B_h)_{i,j} = p^j_{h,i} (0 \leq i \leq 3, 0 \leq j \leq 3)$.

$B_0 = I$,

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k-2 & 0 & 2 & 1 \\ 0 & k-5 & k-8 & k-5 \\ 0 & 2 & 4 & 2 \end{pmatrix}, B_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & k-5 & k-8 & k-5 \\ \frac{(k-2)(k-5)}{2} & \frac{(k-5)(k-8)}{2} & \frac{(k^2-13k+48}{2} & \frac{(k-5)(k-6)}{2} \\ 0 & 2k-10 & 2k-12 & k-5 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2k-10 & 2k-12 & k-5 \\ 2k-4 & 4 & 4 & k-2 \end{pmatrix}.$$

Now we consider an antipodal double cover Γ^* of Γ. The intersection array of Γ^* is the following.

$$\iota(\Gamma^*) = \begin{pmatrix} 0 & 1 & 1 & k-1 & k \\ 0 & 0 & k-2 & 0 & 0 \\ k & k-1 & 1 & 1 & 0 \end{pmatrix}$$

Put $\Omega = \{1, 2, \cdots, k\}$. Let ∞^+ be a vertex of Γ^* and ∞^- be a unique vertex in Γ^* such that $d(\infty^+, \infty^-) = 4$. We may set $\Gamma^*(\infty^+) = \{1^+, 2^+, \cdots, k^+\}$ and $\Gamma^*(\infty^-) = \{1^-, 2^-, \cdots, k^-\}$ and we may consider that $d(i^+, i^-) = 4$ is satisfied for any element $i \in \Omega$. Obviously $\Gamma^*(\infty^+) = \Gamma^*_3(\infty^-)$, $\Gamma^*(\infty^-) = \Gamma^*_3(\infty^+)$ and $\Gamma^*_2(\infty^+) = \Gamma^*_2(\infty^-)$. We denote the subgraph $\Gamma^*_2(\infty^+)$ by Δ and the set of vatices of Δ by X. For each $x \in X$, $|\Gamma^*(\infty^+) \cap \Gamma^*(x)| = 1$ and $|\Gamma^*(\infty^-) \cap \Gamma^*(x)| = 1$ because of $c_2 = b_3 = 1$. Suppose that $|\Gamma^*(\infty^+) \cap \Gamma^*(x)| = \{i^+\}$ and $|\Gamma^*(\infty^-) \cap \Gamma^*(x)| = \{j^-\}$. Then there existes a bijection mapping φ from X onto $(\Omega \times \Omega) \setminus \{(i,i) \mid i \in \Omega\}$ defined by $\varphi(x) = (i,j)$. Then we put $i = \varphi(x)_1$ and $j = \varphi(x)_2$. We denote by x' the element of X such that $d(x, x') = 4$, then $\varphi(x)_1 = \varphi(x')_2$ and $\varphi(x)_2 = \varphi(x')_1$ as we show in the sequal. Moreover we set as follows.

$$A(x) = \{y \in X \mid d(x,y) = 1\},\ B(x) = \{y \in X \mid \varphi(y)_1 = \varphi(x)_2 \text{ or } \varphi(y)_2 = \varphi(x)_1, y \neq x'\}$$

$$A'(x) = \{y \in X \mid d(x',y) = 1\},\ B'(x) = \{y \in X \mid \varphi(y)_1 = \varphi(x)_1 \text{ or } \varphi(y)_2 = \varphi(x)_2, x \neq y\}$$

$$C(x) = X \backslash (A(x) \cup B(x) \cup A'(x) \cup B'(x) \cup \{x, x')$$

We have the following theorem.

Theorem 3.2 *We define relations on X as follows.*

$$R_0 = \{(x,x) \mid x \in X\}, R_1 = \{(x,y) \mid y \in A(x)\}, R_2 = \{(x,y) \mid y \in B(x)\},$$

$$R_3 = \{(x,y) \mid y \in C(x)\}, R_4 = \{(x,y) \mid y \in B'(x)\}, R_5 = \{(x,y) \mid y \in A'(x)\},$$

$$R_6 = \{(x,x') \mid x \in X\}$$

Then $\mathcal{X} = (X, R_i(0 \le i \le 6))$ *is a symmetric association scheme whose parameters are* $p^j_{h,i}(0 \le h, j, i \le 6)$ *in the following matrices.*
Here B_h *is a* 7×7*-matrix whose lows and columns are indexed by* $\{0,1,2,3,4,5,6\}$ *satisfying* $(B_h)_{i,j} = p^j_{h,i}$ *for each* h *such that* $0 \le h \le 6$.

$$B_0 = I, B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ k-2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & k-5 & k-5 & k-8 & k-5 & k-5 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & k-2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 2k-4 & 2 & 1 & 2 & k-3 & 2 & 0 \\ 0 & 2k-10 & k-5 & 2k-12 & k-5 & 2k-10 & 0 \\ 0 & 2 & k-3 & 2 & 1 & 2 & 2k-4 \\ 0 & 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k-5 & k-5 & k-8 & k-5 & k-5 & 0 \\ 0 & 2k-10 & k-5 & 2k-12 & k-5 & 2k-10 & 0 \\ (k-2)(k-5) & (k-5)(k-8) & (k-5)(k-6) & k^2-13k+48 & (k-5)(k-6) & (k-5)(k-8) & (k-2)(k-5) \\ 0 & 2k-10 & k-5 & 2k-12 & k-5 & 2k-10 & 0 \\ 0 & k-5 & k-5 & k-8 & k-5 & k-5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$(B_4)_{i,j} = (B_2)_{i,(6-j)}, (B_5)_{i,j} = (B_1)_{i,(6-j)}, (B_6)_{i,j} = (B_0)_{i,(6-j)} \text{ for } 0 \le i \le 6, 0 \le j \le 6.$$

Proof).
It is immediately shown that
$|A(x)| = |A'(x)| = k-2, |B(x)| = |B'(x)| = 2(k-2)$. We have

$$\{\varphi(x)_1, \varphi(x)_2\} \cap \{\varphi(y)_1, \varphi(y)_2\} = \emptyset \text{ if } d(x,y) = 1 \tag{3.1}$$

from $a_1(\Gamma^*) = 0$, and

$$\{\varphi(y)_1 \mid y \in A(x)\} = \{\varphi(y)_2 \mid y \in A(x)\} = \Omega \setminus \{\varphi(x)_1, \varphi(x)_2\} \tag{3.2}$$

from $c_2(\Gamma^*) = 1$. Hence for any $i \in \Omega \setminus \{\varphi(x)_1, \varphi(x)_2\}$, there exists a unique element $y \in A(x)$ such that $\varphi(y)_1 = i$ and $z \in A(x)$ such that $\varphi(z)_2 = i$ because of $|A(x)| = |\Omega \setminus \{\varphi(x)_1, \varphi(x)_2\}|$.
Therefore the following also holds.

$$(A(y) \setminus \{x\}) \cap (A(z) \setminus \{x\}) = \emptyset \;\; (y \neq z \in A(x)) \tag{3.3}$$

From (3.2) and (3.3), $|\{z \mid z \in A(y), z \neq x, y \in A(x)\}|=(k-2)(k-3)$ and so we have this set is equal to $B(x) \cup C(x)$. Thus $|C(x)| = (k-2)(k-5)$. Moverover we obtain

$$|A(z) \cap B(x)| = 2, \quad |A(z) \cap C(x)| = k-5 \;\; (\forall z \in A(x)) \tag{3.4}$$

$$|A(z) \cap B'(x)| = 2, \quad |A(z) \cap C(x)| = k-5 \;\; (\;\forall z \in A'(x)) \tag{3.5}$$

$$\begin{gathered} |A(z) \cap A(x)| = 1, \quad |A(z) \cap B(x)| = 1, \quad |A(z) \cap C(x)| = k-5 \\ |A(z) \cap B'(x)| = 1 (\;\forall z \in B(x)) \end{gathered} \tag{3.6}$$

$$\begin{gathered} |A(z) \cap A'(x)| = 1, \quad |A(z) \cap B'(x)| = 1, \quad |A(z) \cap C(x)| = k-5, \\ |A(z) \cap B(x)| = 1 (\;\forall z \in B'(x)) \end{gathered} \tag{3.7}$$

$$\begin{gathered} |A(z) \cap A(x)| = 1, \quad |A(z) \cap A'(x)| = 1, \quad |A(z) \cap B(x)| = 2, \\ |A(z) \cap B'(x)| = 2, \quad |A(z) \cap C(x)| = k-8 \;\; (\;\forall z \in C(x)) \end{gathered} \tag{3.8}$$

Moverover about the neighbourhoods of Δ it is easy shown that $\Delta_1(x) = A(x)$, $\Delta_2(x) = B(x) \cup C(x)$, $\Delta_3(x) = B'(x) \cup A'(x)$ and $\Delta_4(x) = \{x'\}$ for any $x \in X$.
About the neighbourhoods of Γ we have $\Gamma_1(x) = A(x) \cup \{\varphi(x)_1^+, \varphi(x)_2^-\}$,
$\Gamma_2(x) = B(x) \cup C(x) \cup B'(x) \cup \{i^+, i^- \mid i \neq \varphi(x)_1, i \neq \varphi(x)_2\} \cup \{\infty^+, \infty^-\}$,
$\Gamma_3(x) = A'(x) \cup \{\varphi(x)_2^+, \varphi(x)_1^-\}$ and $\Gamma_4 = \{x'\}$ for any $x \in X$.
Therefore it follows that $(x, y) \in R_i$ if and only if $(y, x) \in R_i$ for $0 \leq i \leq 6$. We also have $p^j_{h,i} = p^j_{i,h}$ and $p^j_{h,i} = p^{6-j}_{6-h,i}$ since $(x, y) \in R_j$ if and only if $(x', y) \in R_{6-j}$. Then $p^j_{h,i} = p^{6-j}_{i,6-h}$.

Now we prove that $p^3_{3,0} = p^3_{3,6} = 1$, $p^3_{3,1} = p^3_{3,5} = k-8$ and $p^3_{3,2} = p^3_{3,4} = 2k-12$ which means that $p^3_{3,3} = k^2 - 13k + 48$ because of $\sum_{i=0}^{6} p^3_{3,i} = |C(x)| = (k-2)(k-5)$.

It is trivial that $p^3_{3,0} = p^3_{3,6} = 1$. Let x, y be elements of X such that $(x, y) \in R_3$, namely $y \in C(x)$. Then $|C(x) \cap A(y)| = k-8$ from (3.8) and this implies $p^3_{3,1} = k-8$. Considering x' instead of x, similarly above we have $p^3_{3,5} = k-8$. Let z be an element of X such that $(x, z) \in R_3$ and $(z, y) \in R_2$. Set $\varphi(x) = (i, j)$, $\varphi(y) = (k, \ell)$ and $\varphi(z) = (s, t)$, then $\varphi(x') = (j, i)$ and $s = \ell$ or $t = k$ holds because of $(z, y) \in R_2$. Suppose that $s = \ell$ holds. From (3.2) there is a unique element $u \in A(x)$ such that $\varphi(u)_1 = \ell$ and $v \in A'(x)$ such that $\varphi(v)_1 = \ell$. Then we can take any element of Ω except $\{i, j, k, \ell, \varphi(u)_2, \varphi(v)_2\}$ as a number t satisfying $\varphi(z) = (\ell, t)$ and $z \in C(x)$, namely
$|\{t \mid \varphi(z) = (\ell, t),\ (z, y) \in R_2,\ (x, z) \in R_3\}| = k-6$.

Similarly at the case $t = k$, we get
$|\{s \mid \varphi(z) = (s,k),\ (z,y) \in R_2,\ (x,z) \in R_3\}| = k-6$. Hence $p^3_{3,2} = 2(k-6)$ holds. By the same arguments above we have $p^3_{3,4} = 2(k-6)$. Similarly we can decide other parameters $p^j_{h,i}$ from (3.1) $\sim$ (3.8). Thus the theorem is proved.

At the following theorem we prove that the inverse of the statement in theorem 3.2 is also true.

Theorem 3.3 *Let $\mathcal{X} = (X, R_i(0 \le i \le 6))$ be a symmetric 6-association scheme with same parameters as $p^j_{h,i}$ in Theorem 3.2 for $k > 5$. Then the antipodal double cover Γ^* with $d(\Gamma^*) = 4$ of a strongly regular graph with parameters $(k,0,2)$ can be constructed from $\mathcal{X}$. Moreover the graph (X, R_1) is isomorphic to the second neighbourhood of Γ^* with respect to any vertex.*

We now start with a short sketch of the proof. At first we consider the graph $\tilde{\Gamma} = (X, R_4)$. The parameters of this graph is that of the graph deleting the diagonal vertices of $k \times k$-grid. We reconstruct the graph $\hat{\Gamma}$ isomorphic to $k \times k$-grid from $\tilde{\Gamma}$ by adding a set of some pairs of maximal cliques as new vertices to the vertices of $\tilde{\Gamma}$. Next using the graph $\hat{\Gamma}$, an extended graph Γ^* of the graph (X, R_1) is constructed. This Γ^* is the graph to be constructed in this theorem.

We use the following notation here. Let $\Gamma' = (V(\Gamma'), E(\Gamma'))$ be a finite connected graph and d be the metric of Γ'. For two vertices x, y of Γ' such that $d(x,y) = i$, we denote by $c_i(x,y), b_i(x,y)$ and $a_i(x,y)$ the cardinalities of the sets $\{z \in V(\Gamma') \mid d(x,z) = i-1, d(z,y) = 1\}$, $\{z \in V(\Gamma') \mid d(x,z) = i+1, d(z,y) = 1\}$ and $\{z \in V(\Gamma') \mid d(x,z) = i, d(z,y) = 1\}$ respectively.

We state four lemmas to prove the theorem. We note that $k_0 = k_6 = 1, k_1 = k_5 = k-2, k_2 = k_4 = 2k-4$ and $k_3 = (k-2)(k-5)$ hold. Therefore we have $|X| = k(k-1)$. For any element $x \in X$ there exists a unique element $x' \in X$ such that $(x, x') \in R_6$ because of $p^6_{0,6} = 1$. We consider a bijective mapping ψ on X defined by $\psi(x) = x'$ for any $x \in X$. It is clear that $\psi^2 = id_X$. We denote by $\tilde{\Gamma}$ the graph (X, R_4) and by $\tilde{\rho}$ the metric of $\tilde{\Gamma}$.

Lemma 3.1 *The graph $\tilde{\Gamma}$ is a regular graph with the valency $2k-4$ and $d(\tilde{\Gamma}) = 3$. Moreover it follows that $a_1(\tilde{\Gamma}) = k-3$, $b_1(\tilde{\Gamma}) = k-2, a_2(\tilde{\Gamma}) = 2k-6$, $c_2(x,y) = 2$ for $x, y \in X$ such that $\tilde{\rho}(x,y) = 2$ and $y \notin \tilde{\Gamma}(\psi(x))$ and $c_2(x,y) = 1$ for $x, y \in X$ such that $\tilde{\rho}(x,y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. We have also $\tilde{\Gamma}_3(x) = \{\psi(x)\}$ for any $x \in X$.*

Proof).

It is easily verified that $\tilde{\Gamma}$ is a regular graph of the valency $2k-4$ as $p^0_{4,4} = 2k-4$. We take elements $x, y \in X$ such that $(x,y) \in R_i$ for i in $\{1,2,3,4,5\}$. Then $p^i_{4,4} \neq 0$ holds and this implies that there is an element $z \in X$ such that $\tilde{\rho}(x,z) = 1$ and $\tilde{\rho}(z,y) = 1$. Moreover $\tilde{\rho}(x, \psi(x)) = 3$ holds. Therefore we have $d(\tilde{\Gamma}) = 3$ and $\tilde{\rho}(x,y) = 3$ holds if and only if $y = \psi(x)$. Here we note that

$$(x, y) \in R_4 \text{ if and only if } (\psi(x), y) \in R_2. \tag{3.9}$$

because of $p^6_{4,i} = 0$ for any $i (0 \leq i \leq 6, i \neq 2)$ and $p^6_{2,i} = 0$ for any $i (0 \leq i \leq 6, i \neq 4)$. Therefore it follows that

$$\tilde{\rho}(x, y) = 1 \text{ if and only if } \tilde{\rho}(\psi(x), \psi(y)) = 1. \tag{3.10}$$

We now get $a_1(\tilde{\Gamma}) = k - 3$ and $b_1(\tilde{\Gamma}) = k - 2$ because of $p^4_{4,4} = k - 3$ and $\sum_{1 \leq i \leq 5 (i \neq 4)} p^4_{i,4} = k - 2$. For any elements $x, y \in X$ such that $\tilde{\rho}(x, y) = 2$ and $y \notin \tilde{\Gamma}(\psi(x))$, we get $c_2(x, y) = 2, a_2(x, y) = 2k - 6$ and $b_2(x, y) = 0$ because of $p^i_{4,4} = 2, \sum_{1 \leq h \leq 5 (h \neq 4)} p^i_{h,4} = 2k - 6$ for $i = 1, 3$ and 5 and from (3.9). Next let x, y be elements of X such that $\tilde{\rho}(x, y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. Then $(x, y) \in R_2$ from (3.9). We get $c_2(x, y) = 1, b_2(x, y) = 1$ and $a_2(x, y) = 2k - 6$ because of $p^2_{4,4} = p^2_{6,4} = 1$ and $\sum_{1 \leq h \leq 5 (h \neq 4)} p^2_{h,4} = 2k - 6$. We also have $c_3(x, \psi(x)) = 2k - 4$ for any $x \in X$. This completes the proof of the lemma.

Lemma 3.2 *Let x be an element of X. Then $\tilde{\Gamma}(x)$ is a disjoint union of two cliques of the same cadinalities $k - 2$.*

Proof).
Let $x \in X$ and $y \in \tilde{\Gamma}(x)$. Since $a_1(x, y) = k - 3$ and $k(\tilde{\Gamma}) = 2k - 4$ hold, we may set $\tilde{\Gamma}(x) = \{y, y_1, y_2, \cdots, y_{k-3}, z_1, z_2, \cdots, z_{k-2}\}$ for $\{y_1, y_2, \cdots, y_{k-3}\} \subset \tilde{\Gamma}(y)$ and $\{z_1, z_2, \cdots, z_{k-2}\} \subset \tilde{\Gamma}_2(y)$. We set $S = \{y, y_1, y_2, \cdots, y_{k-3}\}$ and $T = \{z_1, z_2, \cdots, z_{k-2}\}$. Let z be any element of T. Then $\tilde{\rho}(y, z) = 2$. Since $c_2(y, z) = 2$, $\tilde{\rho}(x, y) = \tilde{\rho}(x, z) = 1$ and $S \cap \tilde{\Gamma}(z) \subset \tilde{\Gamma}(y) \cap \tilde{\Gamma}(z)$ hold, it follows that $|S \cap \tilde{\Gamma}(z)| \leq 1$. Then we have $|T \cap \tilde{\Gamma}(z)| \geq k - 4$ since $a_1(x, z) = k - 3$. But T contains only $k - 3$ elements except z, therefore $|T \cap \tilde{\Gamma}_2(z)| \leq 1$ holds. Suppose that $T \cap \tilde{\Gamma}_2(z) \neq \emptyset$. Then there exists an element $u \in T$ where $\tilde{\rho}(z, u) = 2$, and every other elements of T except z and u are adjacent to z. Moreover $|T \cap \tilde{\Gamma}_2(u)| \leq 1$ holds as same as $|T \cap \tilde{\Gamma}_2(z)| \leq 1$. Hence we get $T \cap \tilde{\Gamma}_2(u) = \{z\}$. Therefore it follows that x and every elements of T except z and u are contained in $\tilde{\Gamma}(z) \cap \tilde{\Gamma}(u)$, which implies $k - 3 \leq 2$. Thus $k \leq 5$, which contradicts $k > 5$. Hence we get $T \cap \tilde{\Gamma}_2(z) = \emptyset$ and any element of T except z is adjacent to z. However since z is any element of T, T is a clique. Applying the same arguments to a fixed element of T instead of y, we also have S is a clique. Thus the lemma is proved.

We denote by $C_1(x)$ and $C_2(x)$ the set $S \cup \{x\}$ and $T \cup \{x\}$ for S, T in lemma 3.2. We note $|C_1(x)| = |C_2(x)| = k - 1$. Obviously $C_i(x)$ is a maximal clique of $\tilde{\Gamma}$ for $i = 1, 2$ and any maximal clique of $\tilde{\Gamma}$ is equal to $C_i(x)$ for an element $x \in X$ and $i \in \{1, 2\}$. We denote by $MC(\tilde{\Gamma})$ the set of maximal cliques of $\tilde{\Gamma}$ and put $\mathcal{D} = \{C \cup \psi(C) \mid C \in MC(\tilde{\Gamma})\}$. We note that $C \cap \psi(C) = \emptyset$ for any $C \in MC(\tilde{\Gamma})$. For index $i \in \{1, 2\}$ we have $y \in C_i(x)$ if and only if $C_i(x) = C_j(y)$ for some $j \in \{1, 2\}$, as we saw in the proof of lemma 3.2. Hence we have $|MC(\tilde{\Gamma})| = \frac{2|X|}{k-1} = 2k$ and $|\mathcal{D}| = k$. For $i \in \{1, 2\}$ we have $\psi(C_i(x)) = C_j(\psi(x))$

for some $j \in \{1,2\}$ from (3.10). Hence we may put $\psi(C_i(x)) = C_i(\psi(x))$ without loss of generality. We have the following lemma about $\mathcal{D}$.

Lemma 3.3 (1) *Let x be any element of X. Then there exists exactly two elements of $\mathcal{D}$ containing x.*
(2) *Let x, y be any elements of X such that $\tilde{\rho}(x,y) = 1$. Then there exists exactly one element of $\mathcal{D}$ containing x and y.*
(3) *Let x, y be any elements of X such that $\tilde{\rho}(x,y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. Then there exists exactly one element of $\mathcal{D}$ containing x and y.*
(4) *Let D_1 and D_2 be distinct elements of $\mathcal{D}$. Then $|D_1 \cap D_2| = 2$.*
(5) *Let D be an element of $\mathcal{D}$ and x be an element of X such that $x \notin D$. Then $|\tilde{\Gamma}(x) \cap D| = 2$.*

Proof).

(1): For $x \in X$, $C_1(x) \cup \psi(C_1(x))$ and $C_2(x) \cup \psi(C_2(x))$ are distinct elements of $\mathcal{D}$ containing x. Let D be an element of $\mathcal{D}$ such that $x \in D$. Then there is an element $a \in X$ such that $D = C_i(a) \cup \psi(C_i(a))$ for some $i \in \{1,2\}$. We may suppose $x \in C_i(a)$ because of $\psi(C_i(a)) = C_i(\psi(a))$. Then we have $C_i(a) = C_j(x)$ for some $j \in \{1,2\}$ and $D = C_j(x) \cup \psi(C_j(x))$. Thus (1) is proved.

(2): Let x, y be any elements of X such that $\tilde{\rho}(x,y) = 1$. Then there is a unique maximal clique C of $\tilde{\Gamma}$ containing x and y from lemma (3.2). Then $C \cup \psi(C)$ is a unique element of $\mathcal{D}$ containing x and y. Thus (2) is proved.

(3): Let x, y be any elements of X such that $\tilde{\rho}(x,y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. Then $\tilde{\rho}(\psi(x), y) = 1$. Therefore from (2) there exists exactly one element D of $\mathcal{D}$ containing $\psi(x)$ and y. Then obviously $x \in D$ holds. Thus (3) is proved.

(4): Let D_1 and D_2 be distinct elements of $\mathcal{D}$. Then there are elements a and b of X such that $D_1 = C_i(a) \cup \psi(C_i(a))$ and $D_2 = C_j(b) \cup \psi(C_j(b))$ for some $i, j \in \{1,2\}$. We set $\{i, i'\} = \{j, j'\} = \{1,2\}$. We will prove that $D_1 \cap D_2 \neq \emptyset$.

Suppose that $a \in \tilde{\Gamma}(b)$. If $a \in C_j(b)$ or $b \in C_i(a)$, then $D_1 \cap D_2 \neq \emptyset$. Hence we may assume $a \in C_{j'}(b)$ and $b \in C_{i'}(a)$.

Moreover since $\tilde{\rho}(a, \psi(b)) = 2$ and $\tilde{\rho}(\psi(a), \psi(b)) = 1$, there is a unique element $u \in X$, which is adjacent to a and $\psi(b)$ from lemma (3.1). If $u \in C_i(a) \cap C_j(\psi(b))$, then $D_1 \cap D_2 \neq \emptyset$. Hence we may assume $u \in C_{i'}(a)$ or $u \in C_{j'}(\psi(b))$. If $u \in C_{i'}(a)$, then u is adjacent to b because of $b \in C_{i'}(a)$, which means $\tilde{\rho}(b, \psi(b)) = 2$. This is a contradiction. If $u \in C_{j'}(\psi(b))$, then $\psi(u) \in C_{j'}(b)$, then $\psi(u)$ is adjacent to a because of $a \in C_{j'}(b)$, which means $\tilde{\rho}(u, \psi(u)) = 2$. This is also a contradiction. Thus we may assume that a is not adjacent to b. Similarly we may assume a is not adjacent to $\psi(b)$. Hence $\tilde{\rho}(a,b) = 2$ and $\tilde{\rho}(a, \psi(b)) = 2$, and there are exactly two elements $u, v \in X$ which are adjacent to both a and b and there are exactly two elements $u', v' \in X$ which are adjacent to both a and $\psi(b)$ from lemma 3.1. If u is adjacent to v then a is adjacent to b from (2). This contradicts our assumption. Therefore it does not occur that both u and v are contained in one of $\{C_i(a), C_{i'}(a), C_j(b), C_{j'}(b)\}$. For u', v', the same arguments hold. If $u \in C_i(a) \cap C_j(b)$ or $v \in C_i(a) \cap C_j(b)$, then $D_1 \cap D_2 \neq \emptyset$. Hence we may assume that $u \in C_i(a), v \in C_{i'}(a), u \in C_{j'}(b)$ and $v \in C_j(b)$. Similarly we may assume that $u' \in C_i(a), v' \in C_{i'}(a), u' \in C_{j'}(\psi(b))$ and $v' \in C_j(\psi(b))$. Then u and u' are adjacent because of $u, u' \in C_i(a)$ and $\psi(u)$ and u' are adjacent because of $\psi(u), u' \in C_{j'}(\psi(b))$. Therefore we have $\tilde{\rho}(u, \psi(u)) = 2$, which is a contradiction. Thus it follows that $D_1 \cap D_2 \neq \emptyset$.

Now suppose that $C_i(a) \cap C_j(b)$ contains at least two elements u, z. Then from (2) there exists a unique $C \in CM(\tilde{\Gamma})$ containing u, z, and we have $C = C_i(a) = C_j(b)$, which implies $D_1 = D_2$. Therefore $|C_i(a) \cap C_j(b)| \leq 1$.

Similarly $|C_i(a) \cap C_j(\psi(b)| \leq 1$, $|C_i(\psi(a)) \cap C_j(b)| \leq 1$ and $|C_i(\psi(a)) \cap C_j(\psi(b)| \leq 1$. Since $D_1 \cap D_2 = (C_i(a) \cap C_j(b)) \cup (C_i(a) \cap C_j(\psi(b)) \cup (C_i(\psi(a)) \cap C_j(b)) \cup (C_i(\psi(a)) \cap C_j(\psi(b))$, $\psi(C_i(a) \cap C_j(b)) = C_i(\psi(a)) \cap C_j(\psi(b))$ and $\psi(C_i(a) \cap C_j(\psi(b)) = C_i(\psi(a)) \cap C_j(b)$, we have $|D_1 \cap D_2| = 2$, if it is proved that $C_i(a) \cap C_j(b) \neq \emptyset$ is not compatible with $C_i(a) \cap C_j(\psi(b)) \neq \emptyset$.

Suppose that there are elements u, v such that $u \in C_i(a) \cap C_j(b)$ and $v \in C_i(a) \cap C_j(\psi(b))$. Then u and v are adjacent because of $u, v \in C_i(a)$. Moreover $\psi(u)$ and v are adjacent because of $\psi(u), v \in C_j(\psi(b))$. Therefore $\tilde{\rho}(u, \psi(u)) = 2$, a contradiction. Thus (4) is proved.

(5): Let D be an element of $\mathcal{D}$ and y be an element of X such that $y \notin D$. For fix any $j \in \{1, 2\}$, $D \neq C_j(y) \cup \psi(C_j(y))$ because of $y \notin D$. Therefore $|D \cap (C_j(y) \cup \psi(C_j(y))| = 2$ from (4). Hence $|D \cap C_j(y)| = 1$ under consideration $\psi(D) = D$, which means that $|D \cap \tilde{\Gamma}(y)| = 2$. Thus (5) is proved, and the lemma was verified.

We now construct a graph isomorphic to the Hamming graph $H(2, k)$ from $\tilde{\Gamma}$ adding some vertices to X. We define the graph $\hat{\Gamma}$.

The set of vertices of $\hat{\Gamma}$ is $X \cup \mathcal{D}$. The adjacency is defined by $x, y \in X$ are adjacent if $\tilde{\rho}(x, y) = 1$,
$x \in X$ and $D \in \mathcal{D}$ are adjacent if $x \in D$.

The metric of the graph $\hat{\Gamma}$ is denoted by $\hat{\rho}$.

Lemma 3.4 *The graph $\hat{\Gamma}$ is isomorphic to the Hamming graph $H(2, k)$*

Proof).

Let x be any element of X, then there exists exactly two elements of $\mathcal{D}$ containing x and $\psi(x)$. Therefore $\hat{\rho}(x, \psi(x)) = 2$ by the definition above. Hence we have the diameter of $\hat{\Gamma}$ is two. For any $x \in X$, there exists exactly two elements of $\mathcal{D}$ containing x from (1) of lemma 3.3. Moreover, since $k(\tilde{\Gamma}) = 2k - 4$, the valency of x in the graph $\hat{\Gamma}$ is $2k - 2$. For any $D \in \mathcal{D}$, since D contains exactly $2(k - 1)$ elements of X, the valency of D in $\hat{\Gamma}$ is $2k - 2$. Thus the valency of $\hat{\Gamma}$ is $2k - 2$. Let x, y be elements of X such that $\hat{\rho}(x, y) = 1$. Then there exists exactly one element of $\mathcal{D}$ containing x and y from (2) of lemma 3.3.

On the other hand exactly $k-3$ elements of X are adjacent to x and y because of $a_1(\tilde{\Gamma}) = k-3$. Hence it follows $a_1(x,y) = k-2$ in $\hat{\Gamma}$. Let $x \in X$ and $D \in \mathcal{D}$ be adjacent in $\hat{\Gamma}$. Then $x \in D$ and $|D \cap \tilde{\Gamma}(x)| = k-1$. Hence it follows $a_1(x,D) = k-2$ in $\hat{\Gamma}$. Thus $a_1(\hat{\Gamma}) = k-2$ holds. Let x,y be elements of X such that $\hat{\rho}(x,y) = 2$. If $y = \psi(x)$, then obviously $c_2(x,y) = 2$ in $\hat{\Gamma}$. If $y \in \tilde{\Gamma}(\psi(x))$, then there exists exactly one element of $\mathcal{D}$ containing x and y from (3) of lemma 3.3.

Moreover there exists exactly one element of X which is adjacent to x and y because of $c_2(x,y) = 1$ in $\tilde{\Gamma}$ from lemma 3.1. Therefore $c_2(x,y) = 2$ in $\hat{\Gamma}$. If $y \notin \tilde{\Gamma}(\psi(x))$, then there is no element of $\mathcal{D}$ containing x and y since y is not adjacent to x or $\psi(x)$.

However there exists exactly two element of X which are adjacent to x and y because of $c_2(x,y) = 2$ in $\tilde{\Gamma}$. Therefore $c_2(x,y) = 2$ in $\hat{\Gamma}$. Let D_1, D_2 be distinct elements of $\mathcal{D}$. Then $|D_1 \cap D_2| = 2$ from (4) of lemma 3.3. Therefore $c_2(D_1, D_2) = 2$ in $\hat{\Gamma}$. Let D be an element of $\mathcal{D}$ and x be an element of X such that $x \notin D$. Then $|\tilde{\Gamma}(x) \cap D| = 2$ from (5) of lemma 3.3. Therefore $c_2(D,x) = 2$ in $\hat{\Gamma}$. Thus $c_2(\hat{\Gamma}) = 2$ holds. Hence the graph $\hat{\Gamma}$ has the same parameters as those of the Hamming graph $H(2,k)$. Thus the graph $\hat{\Gamma}$ is isomorphic to the Hamming graph $H(2,k)$(cf. [9]). This completes the proof of the lemma.

From lemma 3.4 there exists a bijection φ: $X \cup \mathcal{D} \longmapsto \Omega \times \Omega$ such that $\varphi(\mathcal{D}) = \{(i,i) \mid i \in \Omega\}$ and for any distinct elements $x,y \in X$, $(x,y) \in R_4$ if and only if $\varphi(x)_1 = \varphi(y)_1$ or $\varphi(x)_2 = \varphi(y)_2$ where $\Omega = \{1,2,\cdots,k\}$. We can now construct the antipodal double cover Γ^* of a strongly regular graph with parameters $(k,0,2)$.
The set of vertices of Γ^* is $V(\Gamma^*) = X \cup \Omega^+ \cup \Omega^- \cup \{\infty^{\pm}\}$ where $\Omega^+ = \{1^+, 2^+, \cdots, k^+\}$ and $\Omega^- = \{1^-, 2^-, \cdots, k^-\}$.

The adjacency of Γ^* is defined by
$\Gamma^*(\infty^+) = \Omega^+$, $\Gamma^*(\infty^-) = \Omega^-$; for $x,y \in X$, x and y are adjacent if $(x,y) \in R_1$;
$x \in X$ and $i^+ \in \Omega^+$ are adjacent if $\varphi(x)_1 = i$,
$x \in X$ and $j^- \in \Omega^-$ are adjacent if $\varphi(x)_2 = j$.
The metric of the graph Γ^* is denoted by ρ. Then we get the following.

$$\rho(x,y) = 2 \text{ if } (x,y) \in R_4 \tag{3.11}$$

We can verify that Γ^* is a distance regular graph whose intersection array is $(k, k-1, 1, 1; 1, 1, k-1, k)$ in the sequel. For any $x \in \{\pm\infty\} \cup \Omega^+ \cup \Omega^-$, it is clear that the valency of x is k. For any $x \in X$, there are exactly $k-2$ elements of X which are adjacent to x because of $p^0_{1,1} = k-2$. Moreover x is adjacent to only one element $\varphi(x)^+_1$ in Ω^+ and $\varphi(x)^-_2$ in Ω^- respectively. Therefore the valency of x is k. Thus the valency of Γ^* is k.

We note the bijection φ is a graph isomorphism from $\hat{\Gamma}$ onto the Hamming graph $H(2,k)$ on $\Omega \times \Omega$ such that $\varphi(\mathcal{D}) = \{(i,i) \mid i \in \Omega\}$. Moreover in the subgrph of $H(2,k)$ being deleted the vertices $\{(i,i) \mid i \in \Omega\}$, there exists exactly one vertex at distance 3 from a vertex (i,j) in the subgraph, namely (j,i). This implies the following.

$$\varphi(x) = (i,j) \text{ if and only if } \varphi(\psi(x)) = (j,i) \text{ for } x \in X \tag{3.12}$$

Now we have the following lemma.

Lemma 3.5 *Let x, y be elements of X such that $\varphi(x) = (i,j)$ and $\varphi(y) = (\ell, h)$. Then the following* (1) *and* (2) *hold.*
(1) *If $\rho(x,y) = 1$, then $\{i,j\} \cap \{\ell, h\} = \emptyset$.*
(2) *If $t \in \Omega$ and $t \notin \{i,j\}$, then there exists exactly one element u of X such that $\rho(x,u) = 1$ and $\varphi(u)_1 = t$ and exactly one element v of X such that $\rho(x,v) = 1$ and $\varphi(v)_2 = t$.*

Proof).

(1): Suppose that $\rho(x,y) = 1$. Then $(x,y) \in R_1$. If $i = \ell$ or $j = h$, then $(x,y) \in R_4$, a contradiction. If $i = h$ or $j = \ell$, then $(x, \psi(y)) \in R_4$ from (3.12), therefore $(x,y) \in R_2$ from (3.9), a contradiction. Therefore $\{i,j\} \cap \{\ell, h\} = \emptyset$. Thus (1) holds.

(2):For any distinct elements $u, v \in X$ such that $\rho(x,u) = 1$ and $\rho(x,v) = 1$, we have $\varphi(u)_1 \neq \varphi(v)_1$ and $\varphi(u)_2 \neq \varphi(v)_2$ because of $p^4_{1,1} = 0$.

Moreover since $|\{u \in X \mid \rho(x,u) = 1\}| = k-2$, we have $\Omega = \{\varphi(u)_1 \mid u \in X, \rho(x,u) = 1\} \cup \{i,j\}$ and $\Omega = \{\varphi(u)_2 \mid u \in X, \rho(x,u) = 1\} \cup \{i,j\}$ from (1). Thus (2) holds.

Proof of Theorem 3.3:

Suppose that $x, y \in X$. Since $p^i_{1,1} \neq 0$ for $i \in \{2,3\}$ and $p^5_{1,1} = 0$, the following holds.

$$\rho(x,y) = 2 \text{ if } (x,y) \in R_2 \cup R_3 \tag{3.13}$$

$$\rho(x,y) > 2 \text{ if } (x,y) \in R_5 \tag{3.14}$$

For any $x \in X$, we set as follows.
$A(x) = \{y \in X \mid (x,y) \in R_1\}$,
$B(x) = \{y \in X \mid y \neq \psi(x), \varphi(y)_1 = \varphi(x)_2 \text{ or } \varphi(y)_2 = \varphi(x)_1\}$,
$B'(x) = \{y \in X \mid y \neq x, \varphi(y)_1 = \varphi(x)_1 \text{ or } \varphi(y)_2 = \varphi(x)_2\}$,
$A'(x) = \{y \in X \mid (x,y) \in R_5\}$ and $C(x) = X \setminus (A(x) \cup B(x) \cup B'(x) \cup A'(x) \cup \{\psi(x)\})$.

We note that $y \in B'(x)$ if and only if $(x,y) \in R_4$ and $y \in B(x)$ if and only if $(x,y) \in R_2$ from (3.9) and (3.12). Hence it follows that $y \in C(x)$ if and only if $(x,y) \in R_3$.

Suppose that $x \in X$ and $\varphi(x) = (i,j)$. Then we have $\Gamma^*(x) = A(x) \cup \{i^+, j^-\}$ and $\Gamma^*_2(x) = B(x) \cup C(x) \cup B'(x) \cup (\Omega^+ \setminus \{i^+, j^+\}) \cup (\Omega^- \setminus \{i^-, j^-\}) \cup \{\infty^\pm\}$ from (3.11),(3.13) and (2) of Lemma 3.5.

Moreover obviously $A(y) \cap \Gamma^*_2(x) \neq \emptyset$ for any $y \in A'(x)$. Hence we have $\Gamma^*_3(x) = A'(x) \cup \{i^-, j^+\}$ from (3.14) and $\Gamma^*_4(x) = \{\psi(x)\}$. On the other hand for any $i \in \Omega$, we have $\Gamma^*(i^+) = \{x \in X \mid \varphi(x)_1 = i\} \cup \{\infty^+\}$, $\Gamma^*_2(i^+) = \{x \in X \mid \varphi(x)_1 \neq i$ and $\varphi(x)_2 \neq i\} \cup (\Omega^+ \setminus \{i^+\}) \cup (\Omega^- \setminus \{i^-\})$, $\Gamma^*_3(i^+) = \{x \in X \mid \varphi(x)_2 = i\} \cup \{\infty^-\}$ and $\Gamma^*_4(i^+) = \{i^-\}$. Therefore especially it follows that the diameter of Γ^* is 4.

Now since $p^i_{1,6} = 0$ for $i \in \{0,1,2,3,4,6\}$ and $p^i_{5,6} = 0$ for $i \in \{0,2,3,4,5,6\}$, we obtain the following.

$$(x,y) \in R_1 \text{ if and only if } (x, \psi(y)) \in R_5 \tag{3.15}$$

This statement with (3.9) and (3.12) imply that $\Gamma^*(x) = \Gamma_3^*(\psi(x))$, $\Gamma_2^*(x) = \Gamma_2^*(\psi(x))$ and $\Gamma_3^*(x) = \Gamma^*(\psi(x))$ for any $x \in X$. Therefore we have $c_1(\Gamma^*) = b_3(\Gamma^*)$, $c_2(\Gamma^*) = b_2(\Gamma^*)$, $c_3(\Gamma^*) = b_1(\Gamma^*)$ and $c_4(\Gamma^*) = b_0(\Gamma^*)$.

Lastly we will prove that $a_1(\Gamma^*) = 0$ and $c_2(\Gamma^*) = 1$, which lead to a complete proof of Theorem 3.3. Since $p_{1,1}^1 = 0$, there are no triangle whose vertices are all in X. Moreover for any elements $x, y \in X$ such that $\rho(x,y) = 1$ it follows that $\psi(x)_1 \neq \psi(y)_1$ and $\psi(x)_2 \neq \psi(y)_2$ from (1) of Lemma 3.5. Thus there are no triangle in Γ^*. Hence we have $a_1(\Gamma^*) = 0$ and $b_1(\Gamma^*) = k-1$.

Let x, y be elements in X and suppose that $\rho(x,y) = 2$. Then $y \in B(x) \cup C(x) \cup B'(x)$. If $y \in B(x) \cup C(x)$, then $c_2(x,y) = 1$ because of $p_{1,1}^2 = 1$ and $p_{1,1}^3 = 1$. If $y \in B'(x)$, then $c_2(x,y) = 1$ because $p_{1,1}^4 = 0$ and either $\varphi(x)_1 = \varphi(y)_1$ or $\varphi(x)_2 = \varphi(y)_2$ occurs.

Next suppose that $\rho(x, i^+) = 2$ for $x \in X$ and $i \in \Omega$. Then from (2) of Lemma 3.5, we have $c_2(x, i^+) = 1$. Obviously $c_2(\infty^+, x) = 1$ for any $x \in X$, ,$c_2(i^+, j^+) = 1$ for any distinct $i, j \in \Omega$ and $c_2(i^+, j^-) = 1$ for any distinct $i, j \in \Omega$. Thus it is proved that $c_2(\Gamma^*) = 1$. This completes the proof of the theorem.

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